Geometrical SO(4, 1) gauge theory as a basis of extended relativistic objects for hadrons

1990 J. Phys. A: Math. Gen. 23 1885

(http://iopscience.iop.org/0305-4470/23/11/016)

J. Phys. A: Math. Gen. 23 (1990) 1885-1907. Printed in the UK

Geometrical SO(4, 1) gauge theory as a basis of extended relativistic objects for hadrons

R R Aldinger
Department of Physics, Gettysburg College, Gettysburg, PA 17325, USA

Received 14 November 1989

Abstract. The geometrical structure for a collective model description of extended relativistic single hadron systems is investigated by considering a Yang-Mills theory of the de Sitter group SO(4, 1). The full SO(4, 1) symmetry is broken down to that of its SO(3, 1) stability subgroup resulting in a set of Goldstone fields which are taken to represent coordinates of a point in the de Sitter fibre space and are used, along with the original linear gauge fields, to define the vierbein and spin connection on the restricted bundle: $P'(\mathfrak{M}_4, SO(3,1)) \subset P(\mathfrak{M}_4, SO(4,1))$. The symmetry breaking parameter is taken as a fundamental length relevant to hadron physics. The original linear gauge fields generate a type of parallel transport which is the curved space analogue of development into the flat affine tangent space and serves as the bridge between the geometrical and purely gauge-theoretic descriptions. Upon quantisation, the generator of development in the unitary gauge (Higgs mechanism), for a special class of horizontal Lorentz cross sections, goes over into the de Sitter space momentum which serves to break the mass-spin degeneracy inherent in the Poincaré group description and supplies a curved space perturbation in the resulting relativistic Hamiltonian. The Hamiltonian is used to determine a completely solvable set of dynamical equations of motion resulting in the *Zitterbewegung* of the extended relativistic object.

1. Introduction

Strongly interacting particles are classified using the internal symmetry groups while their spacetime related attributes are typically gauged by using symmetries of the physical Poincaré group. However, any Poincaré gauge model concerned with the geometrical aspects of elementary particles will inherit the well known drawback associated with the original attempts at obtaining a gauge-theoretical interpretation of general relativity [1-5]. Namely, that only the local Lorentz invariance is realised as an ordinary gauge invariance while local translations are absorbed by invariance under general coordinate transformations. As a result, the fundamental underlying concept of gauging using specific local symmetry transformations does not apply and the complete Yang-Mills picture is not maintained.

However, a general procedure has been laid out that eliminates the difficulties associated with earlier attempts to formulate a gauge theory of gravity in which explicit gauge invariance is obtained as a consequence of the spontaneous breaking of some larger symmetry [6-11]. For example, one may consider the physical Poincaré group as the contraction limit of one of the de Sitter (dS) groups leading to a spontaneously broken Yang-Mills theory in which the symmetry-breaking parameter is the radius of the dS space where the direction of the breaking is specified by the usual Higgs-type

0305-4470/90/111885+23$03.50

mechanism. Consequently, translational gauge invariance and invariance under general coordinate transformations are kept separate and mix only after the Higgs field has been 'fixed' in some prescribed direction (choice of gauge).

Therefore it is possible to retain the complete gauge picture throughout without absorbing local translations as long as each Minkowski tangent space is replaced with a non-compact homogeneous dS space (characterised by the fundamental length parameter, R) such that when the strength of the symmetry-breaking tends to infinity, the dS group goes over into the physical Poincaré group and each dS space becomes a typical Minkowski space. In the contraction process, the four dimensionless parameters of the dS group are scaled by a factor of R and become the translation parameters of ISO(3, 1).

In this paper we make use of this formalism in order to investigate the insights into the internal spacetime sector of chargeless non-interacting single hadron systems that can be gained by considering a spontaneously broken Yang–Mills theory of the dS group SO(4, 1). More specifically, the questions concerning a spacetime related non-Abelian gauge field description of a geometrically motivated mechanism for non-locality-confinement, a completely solvable quantum relativistic dynamics of non-local microstructures, and an associated experimentally verifiable mass–spin trajectory relation are addressed by complementing the constituent quark–gluon substructure (the 'fast' variables) of the conventional colour gauge field theory with a collective model ('slow' variable) geometry realised on a soldered (Cartan-type) [12] dS fibre bundle erected over a strongly curved but gravitationally flat four-dimensional spacetime.

That is, we adopt Drechsler's [13, 14] original idea as our fundamental physical picture for a geometrical gauge theory of strongly interacting particles and ignore any long-range gravitational fields in order to concentrate exclusively on the short-range (hadronically-induced) affects on spacetime by assuming that the local non-flat character of the underlying spacetime in the immediate vicinity of an attached micro-dS space (with radius on the order of one fermi) can be directly attributed to the hadronically-induced solder mechanism which essentially allows one to associate the physically observable spacetime imprint with the internal dS fibre space. Therefore we consider the internal dS group SO(4, 1) as a spacetime related symmetry group while establishing a short-range modification of the conventional Minkowski geometry (typically associated with the arena for the Poincaré model of structureless elementary point particle kinematics) leading to a curved-space gauge theory for a collective model description of relativistically extended hadronic structures valid in the strongly interacting regime ($\approx$1 GeV) of QCD.

The organisation of the paper is as follows. In section 2, we review the details of an SO(4, 1) gauge theory. The pullback of the Cartan connection in $P(\mathfrak{M}_4, SO(4,1))$ to $P'(\mathfrak{M}_4, SO(3,1)) \subset P(\mathfrak{M}_4, SO(4,1))$ is emphasised. The Goldstonians of the symmetry-breaking mechanism are taken to represent coordinates of a point in dS space, $\Sigma_x^4 = SO(4,1)/SO(3,1)_x$, and, using the usual group theoretical techniques of nonlinear realisations [15–17], are used along with the original linear gauge fields to define the vierbein and spin connection [18, 19]. A description of the complete geometrical picture (concerning the effects of both spacetime curvature and torsion) is taken up in section 3 by considering a curved space generalisation of the operator of development [20] along with its associated notion of parallel transport *à la* Stelle and West in [8]. Also discussed in this section are certain gauge-fixing relations which serve to simplify the full gauge-theoretic description thereby allowing one to make contact with the model of the quantum relativistic rotator, QRR [21-25] (which has met with much success in describing the rotational aspects of single hadron systems). Quantisation of the SO(4, 1) gauge theory is carried out in section 4 where we employ the usual gauge concept of particle interactions for a relativistic theory by interpreting the SO(4, 1) generator of development as the SO(4, 1) 'generalised momentum' in which the pseudo translational piece supplies an experimentally detectable perturbation to the conventional Minkowskian flat-space description. A quantum relativistic Hamiltonian is obtained by applying the usual techniques of Dirac's constrained Hamiltonian mechanics [26] where the constraint relation leads to an experimentally verifiable (rotator-like) mass-spin trajectory relation. In section 5 we derive a set of solvable dynamical equations of motion which leads to the *Zitterbewegung* of the extended object. Finally, the main results obtained in this paper are presented in section 6.

2. SO(4, 1) gauge theory and Higgs mechanism

Consider the vacuum state of spacetime to be four-dimensional pseudo Riemannian with constant curvature of radius R and possessing a dS SO(4, 1) as its global symmetry group of transformations. The four-dimensional manifold with constant curvature can be parametrised by five coordinates Φ_A which for SO(4, 1) are constrained according to:

$$\Phi^A\Phi^B\eta_{AB} = -R^2 \qquad A, B = 0, 1, 2, 3, 5$$

where the dS metric $\eta_{AB} = \mathrm{diag}(1, -1, -1, -1, -1)$. The dS Lie algebra G is generated by the $R_{4,1}$ angular momenta $J_{AB} = -J_{BA}$ which satisfy the commutation relation

$$[J_{AB}, J_{CD}] = -\mathrm{i}(\eta_{AD}J_{BC} - \eta_{BC}J_{AD} + \eta_{BD}J_{AC} - \eta_{AC}J_{BD}). \tag{2.1}$$

Given some specific point of coordinates $\underset{0}{\Phi}{}^A$ the SO(4, 1) generators can be decomposed as [6]

$$J_{AB} = J_{ij} + \underset{0}{\Phi}{}_A\pi_j + \underset{0}{\Phi}{}_B\pi_i \tag{2.2a}$$

where

$$J_{ij} = J_{AB} - R^{-2}(\underset{0}{\Phi}{}_A\underset{0}{\Phi}{}^C J_{CB} - \underset{0}{\Phi}{}_B\underset{0}{\Phi}{}^C J_{AC}) \tag{2.2b}$$

and

$$\pi_j = R^{-2}\underset{0}{\Phi}{}^A J_{AB} \tag{2.2c}$$

with $i, j = 0, 1, 2, 3$. That is, the i, j have components only in the orthogonal subspace to $\underset{0}{\Phi}{}^A$ such that: $\underset{0}{\Phi}{}^A J_{ij} = \underset{0}{\Phi}{}^A\pi_i = 0$. The J_{ij} generate the SO(3, 1) stability subgroup of $\underset{0}{\Phi}{}^A$ (keeping $\underset{0}{\Phi}{}^A$ fixed while rotating its tangent space) and the π_i are generators of infinitesimal translation of $\underset{0}{\Phi}{}^A$. Equation (2.2a) leads to the usual four-dimensional realisation of SO(4, 1):

$$[J_{ij}, J_{kl}] = -\mathrm{i}(\eta_{il}J_{jk} - \eta_{jk}J_{il} + \eta_{jl}J_{ik} - \eta_{ik}J_{jl}) \tag{2.3a}$$

$$[J_{ij}, \pi_k] = -\mathrm{i}(\eta_{ik}\pi_j - \eta_{ij}\pi_k) \tag{2.3b}$$

$$[\pi_i, \pi_j] = \mathrm{i}R^{-2}J_{ij} \tag{2.3c}$$

where the SO(3, 1) metric $\eta_{ij} = \mathrm{diag}(1, -1, -1, -1)$.

The structure constants f^C_{AB} of G have the form

$$f^{mn}_{ij,kl} = \eta_{ik}\delta^m_j\delta^n_l + \eta_{jl}\delta^m_i\delta^n_k - \eta_{il}\delta^m_j\delta^n_k - \eta_{jk}\delta^m_i\delta^n_l$$

$$f^m_{i,jk} = \eta_{ik}\delta^m_j - \eta_{ij}\delta^m_k$$

$$f^{mn}_{ij} = -\left(\frac{1}{R^2}\right)(\delta^m_i\delta^n_j - \delta^m_j\delta^n_i).$$

Therefore the Cartan metric tensor, $C_{AB} = f^D_{AC} f^C_{BD}$, may be expressed according to [27]

$$C = \begin{bmatrix} C_{ij,kl} & C_{ij,k} \\ C_{k,ij} & C_{ij} \end{bmatrix} = \begin{bmatrix} 6(\eta_{il}\eta_{jk} - \eta_{jl}\eta_{ik}) & 0 \\ 0 & (6/R^2)\eta_{ij} \end{bmatrix}.$$

To allow for local group transformations we now associate to each point x^μ of the underlying spacetime manifold $\mathfrak{M}_4$ (no longer in the vacuum configuration) an internal space $\Sigma^4_x = \mathrm{SO}(4,1)/\mathrm{SO}(3,1)_x$ which is a local copy of the vacuum (one-sheeted hyperboloid non-compact in time and compact in the space directions). The union of fibres, Σ^4_x, represents the dS fibre bundle: $E = \cup\Sigma^4_x$ where Σ^4_x for every $x \in \mathfrak{M}_4$ is related to the typical fibre Σ^4 by a map: $\Sigma^4_x \Rightarrow \Sigma^4$. The dS bundle $E(\mathfrak{M}_4, \Sigma^4, \mathrm{SO}(4,1), P)$ is associated to the principal dS (frame) bundle: $P(\mathfrak{M}_4, \mathrm{SO}(4,1))$. On each Σ^4_x we have a point $\Phi^A_x(x)$ which determines which $\mathrm{SO}(3,1)_x$ subgroup of SO(4, 1) is going to be the physical Lorentz group at each $x_\mu \in \mathfrak{M}_4$. That is, $\Phi^A_x(x)$ is the Higgs field which determines the 'direction' of the symmetry breakdown: SO(4, 1) → SO(3, 1).

The pullback of the Cartan connection in P to $\mathfrak{M}_4$, $\tilde{\Gamma}_\mu(x)$, specifies the transport of some $\mathrm{SO}(4,1)_x$-vector into an $\mathrm{SO}(4,1)_{x+\mathrm{d}x}$-vector and thereby determines the nature of a local dS frame when one performs an infinitesimal transformation in the μ direction on $\mathfrak{M}_4$ where

$$\tilde{\Gamma}_\mu(x) = \tfrac{1}{2}\tilde{\Gamma}_\mu{}^{AB}(x) J_{AB}. \tag{2.4}$$

The gauge potentials $\{\tilde{\Gamma}_\mu{}^{AB}\}$ represent the 40 dS rotation coefficients of the Cartan connection. The SO(4, 1) covariant derivative of Φ^A is

$$\tilde{\nabla}_\mu \Phi^A = \partial_\mu \Phi^A + \tilde{\Gamma}_\mu{}^A{}_C \Phi^C \tag{2.5}$$

and parallel transport about a small closed curve on $\mathfrak{M}_4$ results in the SO(4, 1) curvature of the Cartan connection in P:

$$[\tilde{\nabla}_\mu, \tilde{\nabla}_\nu] = \mathrm{i}\tfrac{1}{2}\tilde{\mathfrak{R}}_{\mu\nu}{}^{AB} J_{AB}$$

with curvature coefficients (gauge fields)

$$\tilde{\mathfrak{R}}_{\mu\nu}{}^{AB} = \partial_\mu \tilde{\Gamma}_\nu{}^{AB} - \partial_\nu \tilde{\Gamma}_\mu{}^{AB} + \tilde{\Gamma}_\mu{}^{AC}\tilde{\Gamma}_{\nu C}{}^B - \tilde{\Gamma}_\nu{}^{AC}\tilde{\Gamma}_{\mu C}{}^B. \tag{2.6}$$

Gauge invariance of the covariant derivative implies the typical inhomogeneous transformation character of the $\tilde{\Gamma}_\mu$

$$\tilde{\Gamma}'_\mu = g\tilde{\Gamma}_\mu g^{-1} - \mathrm{i} g \partial_\mu g^{-1}$$

where $g \in G = \mathrm{SO}(4,1)$ while the Cartan curvature field transforms homogeneously as expected

$$\tilde{\mathfrak{R}}'_{\mu\nu} = g\tilde{\mathfrak{R}}_{\mu\nu} g^{-1}.$$

The SO(4, 1) Lie algebraic decomposition implied by (2.2*a*) corresponds to: G = H + T where H is the subalgebra generating the stability subgroup $H = \mathrm{SO}(3,1)$ of $G = \mathrm{SO}(4,1)$ and T is a four-dimensional vector subspace $R_{3,1}$ of G spanning the tangent space $T_\Phi(\Sigma_x^4)$ to Σ_x^4 at the specified point Φ. Accordingly, the Cartan connection defined on P is restricted to the principle bundle $P' = P'(\mathfrak{M}_4, \mathrm{SO}(3,1)) \subset P(\mathfrak{M}_4, \mathrm{SO}(4,1))$ (where there exists an injective map $\gamma: P' \to P$) leading to the pullback of $\tilde{\Gamma}_\mu$ to P': $\Gamma_\mu = \gamma^* \tilde{\Gamma}_\mu$ such that

$$\Gamma_\mu = \omega_\mu - \theta_\mu = \tfrac{1}{2}\omega_\mu{}^{ij} J_{ij} - \theta_\mu{}^i \pi_i \tag{2.7a}$$

with

$$\omega_\mu{}^{ij} = \Gamma_\mu{}^{ij} \tag{2.7b}$$

and

$$\theta_\mu{}^i = R\Gamma_\mu{}^{5i} \tag{2.7c}$$

(which corresponds to the (pseudo) translational gauge connection). Furthermore, the pullback of the Cartan field strength (2.6) decomposes according to

$$\tfrac{1}{2}\mathfrak{R}_{\mu\nu}{}^{AB} J_{AB} = \tfrac{1}{2} Q_{\mu\nu}{}^{ij} J_{ij} - S_{\mu\nu}{}^i \pi_i \tag{2.8a}$$

where

$$Q_{\mu\nu}{}^{ij} = R_{\mu\nu}{}^{ij} + R^{-2}(\theta_\mu{}^i \theta_\nu{}^j - \theta_\nu{}^i \theta_\mu{}^j) \tag{2.8b}$$

$$R_{\mu\nu}{}^{ij} = \partial_\mu \omega_\nu{}^{ij} - \partial_\nu \omega_\mu{}^{ij} + \omega_\mu{}^{ik} \omega_{\nu k}{}^j - \omega_\nu{}^{ik} \omega_{\mu k}{}^j \tag{2.8c}$$

$$S_{\mu\nu}{}^i = \partial_\mu \theta_\nu{}^i - \partial_\nu \theta_\mu{}^i + \omega_\mu{}^i{}_k \theta_\nu{}^k - \omega_\nu{}^i{}_k \theta_\mu{}^k. \tag{2.8d}$$

The field equations for $\tilde{\Gamma}_\mu{}^{AB}$ are obtained by considering the Yang-Mills Lagrangian density [28]:

$$L = -(\kappa/2) C_{AB} \tilde{\Omega}^A \wedge *\tilde{\Omega}^B \tag{2.9}$$

where κ is a coupling constant, $\tilde{\Omega}^A$ is the curvature two-form

$$\tilde{\Omega}^A = \tfrac{1}{2}\tilde{\mathfrak{R}}_{\mu\nu}^A \, \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu$$

of the Cartan connection and the Hodge $*$ operator defines the dual form $*\tilde{\Omega}^B$.

In order to specify the Lagrangian density any further, one must first properly interpret the linear gauge potentials: $\{\omega_\mu{}^{ij}, \theta_\mu{}^i\}$. As a first step, we note that the decomposition of the pullback Γ_μ into SO(3, 1) and $R_{3,1}$ components given in (2.7) is clearly no longer ᴅS gauge invariant but now holds invariance only under the subset of SO(3, 1) transformations. Thus it seems as though we do not possess any means of parallel transporting ordinary Lorentz four-vectors while retaining the complete ᴅS symmetry throughout. However, under the process of spontaneous symmetry breakdown, four of the original ten ᴅS generators break allowing the passage to a nonlinear realisation of SO(4, 1) on the homogeneous non-compact coset space, Σ^4 (in which the nonlinearly transforming sets of SO(4, 1) fields transform independently according to their SO(3, 1) index type). It is therefore necessary to determine the induced nonlinear gauge fields on $E(\mathfrak{M}_4, \Sigma^4, \mathrm{SO}(4,1), P)$ where the ᴅS fibre $\Sigma^4 = \mathrm{SO}(4,1)/\mathrm{SO}(3,1)_0$ (with the fixed point 0 in $R_{4,1} \supset \Sigma^4$) is the space of nonlinear realisations of SO(4, 1).

We begin by determining some convenient parametrisation of the point $\Phi(x) \in \Sigma_x^4$. Let $\underset{0}{\Phi}$ be a fixed element of Σ^4 and J_{ij} be the generators of the stability subgroup

$H_0 = SO(3,1)_0$ of $\underset{0}{\Phi}$ such that: $J_{ij}\underset{0}{\Phi}^A = 0$. The arbitrary point $\Phi(x)$ is reached from $\underset{0}{\Phi}(x)$ by acting with any element g of $G = SO(4,1)$ according to the prescription:

$$\Phi(x) = g\underset{0}{\Phi}(x) \qquad g \in G = SO(4,1).$$

Furthermore, for any element g there exists a unique decomposition

$$g = g_F g_H \qquad g_F \in F = G/H_0 = SO(4,1)/SO(3,1)_0 \qquad g_H \in H_0 = SO(3,1)_0$$

where $g_H \underset{0}{\Phi}(x) = 0$. In the standard exponential parametrisation scheme

$$g_F = \exp(-\mathrm{i}\xi \cdot \pi)$$

such that the functions $\xi^i(x)$ completely determine the point $\Phi(x)$.

The original linear gauge fields $\{\Gamma_\mu{}^{AB}\} = \{\omega_\mu{}^{ij}, \theta_\mu{}^i\}$ are used to induce the nonlinear gauge fields on $E(\mathfrak{M}_4, \Sigma^4, SO(4,1), P)$ by applying the group-theoretical techniques of nonlinear realisations such that:

$$\begin{aligned} \mathrm{i}\bar{\Gamma}_\mu &= \mathrm{i}\tfrac{1}{2}\bar{\Gamma}_\mu{}^{AB}J_{AB} = \mathrm{i}\tfrac{1}{2}\bar{\omega}_\mu{}^{ij}J_{ij} - \mathrm{i}\bar{\theta}_\mu{}^i\pi_i \\ &= \exp(\mathrm{i}\xi\cdot\pi)[\partial_\mu + \mathrm{i}\tfrac{1}{2}\omega_\mu{}^{ij}J_{ij} - \mathrm{i}\theta_\mu{}^i\pi_i]\exp(-\mathrm{i}\xi\cdot\pi). \end{aligned} \tag{2.10}$$

The induced gauge fields $\{\bar{\Gamma}_\mu{}^{AB}\} = \{\bar{\omega}_\mu{}^{ij}, \bar{\theta}_\mu{}^i\}$ represent what are known as the physical fields of a spontaneously broken gauge field theory and are explicitly expressed as rather complex functions of the linear gauge fields, coset parameters, and their derivatives such that:

$$\bar{\omega}_\mu{}^{ij} = \omega_\mu{}^{ij} + \overset{(0)}{\omega}{}_\mu{}^{ij} + \check{\omega}_\mu{}^{ij} \tag{2.11a}$$

$$\bar{\theta}_\mu{}^i = \theta_\mu{}^i + \overset{(0)}{\theta}{}_\mu{}^i + \check{\theta}_\mu{}^i \tag{2.11b}$$

where $\overset{(0)}{\omega}{}_\mu{}^{ij}$ and $\overset{(0)}{\theta}{}_\mu{}^i$ denote those parts of the forms which are independent of the original gauge fields and depend upon the coset parameters and their derivatives only (the 'classical' parts of the nonlinear gauge fields) while $\check{\omega}_\mu{}^{ij}$ and $\check{\theta}_\mu{}^i$ are proportional to the corresponding linear gauge fields and, along with $\omega_\mu{}^{ij}$ and $\theta_\mu{}^i$, can be chosen as new independent variables. In the so-called 'unitary gauge' choice where $\xi^i = (0,0,0,0)$, $\bar{\omega}_\mu{}^{ij} = \omega_\mu{}^{ij}$ and $\bar{\theta}_\mu{}^i = \theta_\mu{}^i$. The classical part of the vierbein defines the maximally flat (background) spacetime which has the isometry group SO(4, 1).

In terms of stereographic projection coordinates, Z^i, on Σ^4 (projection of the dS hyperboloid on the tangent hyperplane $T_\xi(\Sigma^4_x)$ at the point ξ) where

$$\xi^i(x) = \frac{Z^i(x)}{1 - Z^2/4R^2} \qquad \xi^5 = -R\frac{1 + Z^2/4R^2}{1 - Z^2/4R^2}$$

with $Z^2 = Z^iZ_i = (Z^0)^2 - (Z^1)^2 - (Z^2)^2 - (Z^3)^2$, we have that

$$\overset{(0)}{\omega}{}_\mu{}^{ij}(x) = \tfrac{1}{2}R^{-2}F(x)(Z^i\partial_\mu Z^j - Z^j\partial_\mu Z^i) \tag{2.12a}$$

and

$$\overset{(0)}{\theta}{}_\mu{}^i(x) = F(x)\delta_\mu{}^i \tag{2.12b}$$

where

$$F(x)=\frac{1}{1-Z^2/4R^2}.$$

The line element has the form

$$\mathrm{d}s^2=\overset{(0)}{g}_{\mu\nu}\,\mathrm{d}Z^\mu\,\mathrm{d}Z^\nu=\eta_{\mu\nu}\frac{\mathrm{d}Z^\mu\,\mathrm{d}Z^\nu}{(1-Z^2/4R^2)^2}$$

where the spacetime metric tensor

$$\overset{(0)}{g}_{\mu\nu}=\overset{(0)}{\theta}_\mu{}^i\,\overset{(0)}{\theta}_{\nu i}=\eta_{\mu\nu}[F(x)]^2.$$

Under local SO(4, 1) gauge transformations, the induced fields transform according to a representation of the SO(3, 1) stability subgroup

$$-\mathrm{i}\bar{\theta}'_\mu{}^i\pi_i=g_{H_1}(-\mathrm{i}\bar{\theta}_\mu{}^i\pi_i)g_{H_1}^{-1} \tag{2.13a}$$

$$\mathrm{i}\tfrac{1}{2}\bar{\omega}'_\mu{}^{ij}J_{ij}=g_{H_1}(\mathrm{i}\tfrac{1}{2}\bar{\omega}_\mu{}^{ij}J_{ij})g_{H_1}^{-1}+g_{H_1}\partial_\mu g_{H_1}^{-1} \tag{2.13b}$$

where $g_{H_1}\in H_0=\mathrm{SO}(3,1)_0$ is a (nonlinear) function of ξ^i and $g\in G=\mathrm{SO}(4,1)$. Therefore $\bar{\omega}_\mu{}^{ij}$ transforms as a typical connection while $\bar{\theta}_\mu{}^i$ transforms homogeneously (characteristic of a nonlinear gauge field) according to the four-vector representation of SO(3, 1) but with the nonlinear element g_{H_1}.

Since the SO(4, 1) translations are broken, the $\bar{\theta}_\mu{}^i$ should not be considered as true translational gauge degrees-of-freedom but are instead of a set of four-vector fields (vector bosons) with the transformation character expressed by (2.13*a*). Of course we still retain the freedom in making some SO(3, 1) gauge choice and, in a certain sense, we have reduced the full dS gauge theory down to a Lorentz gauge subtheory. However, in order to make the reduction to an SO(3, 1) gauge theory complete [29], it is necessary to go one step further and to eliminate the $\bar{\theta}_\mu{}^i$ altogether which can be done quite naturally by identifying it with the spacetime vierbein $\bar{h}_\mu{}^i$ (the soldering condition):

$$\bar{\theta}_\mu{}^i=\bar{h}_\mu{}^i \qquad \text{(solder mechanism)} \tag{2.14}$$

(the components of which are given *a priori* and therefore cannot participate in the description of any specific gauge field description).

So, the Higgs mechanism enhanced by soldering allows one to interpret the nonlinear gauge fields $\{\bar{\omega}_\mu{}^{ij},\bar{\theta}_\mu{}^i\}$, expressed by (2.11*a*) and (2.11*b*) as the spacetime spin and vierbein fields $\{\bar{\omega}_\mu{}^{ij},\bar{h}_\mu{}^i\}$ where the components, $\bar{h}_\mu{}^i$, of the soldering form provide an isomorphism between the tangent space to spacetime at $x\in\mathfrak{M}_4$: $T_x(\mathfrak{M}_4)$ and the tangent space to Σ_x^4 at $\xi^i(x)\in\Sigma_x^4$: $T_{\xi(x)}(\Sigma_x^4)$, where the $\xi^i(x)$ are selected out by the cross section (choice of gauge).

Under the soldering condition, the Σ_x^4 metric η_{ij} uniquely determines a metric on $\mathfrak{M}_4$ thereby establishing an association between the strictly spacetime related features based on the $g_{\mu\nu}$ and the attached internal dS space with group metric η_{ij} where:

$$g_{\mu\nu}=\bar{h}_\mu{}^i\bar{h}_\nu{}^j\eta_{ij}. \tag{2.15}$$

Consequently, the structural group $G=\mathrm{SO}(4,1)$ can be interpreted as acting on either the vacuum images, Σ_x^4, or on spacetime itself. Not only does the soldering mechanism between $T_x(\mathfrak{M}_4)$ and $T_{\xi(x)}(\Sigma_x^4)$ provide a metric on $\mathfrak{M}_4$ but it also supplies a means of measuring distance in the attached fibre space. That is, without soldering, the radius R of the internal dS space Σ_x^4 has no direct physical implication. Distance has meaning only on $\mathfrak{M}_4$ where it is measured in metres and soldering transfers the meaning of distance to the internal Σ_x^4 so that R is, once again, measured in units of metres.

Using the decomposition of the Cartan field strength (2.8) along with (2.14) and the explicit form of the Cartan metric, C_{AB}, allows the Lagrangian density (2.9) to be expressed in terms of the connection and solder forms in the bundle $P'(\mathfrak{M}_4, \mathrm{SO}(3,1))$. Up to the invariant volume element we have that (see Zardecki [28]):

$$L = 3\kappa[(1/2R^2)\bar{R}_{ij\mu\nu}\bar{h}^{i\mu}\bar{h}^{j\nu} + \tfrac{1}{4}\bar{R}_{ij\mu\nu}\bar{R}^{ij\mu\nu} + (1/2R^4)g^{\mu\lambda}g^{\nu\sigma}(\eta_{il}\eta_{jk} - \eta_{jl}\eta_{ik})\bar{h}^i_{\ \mu}\bar{h}^j_{\ \nu}\bar{h}^k_{\ \lambda}\bar{h}^l_{\ \sigma} - (1/2R^2)\bar{S}_{i\mu\nu}\bar{S}^{i\mu\nu}]$$

where the SO(3, 1) curvature, $\bar{R}_{\mu\nu}{}^{ij}$, is given by (2.8*c*) with (2.11*a*) and the torsion, $\bar{S}_\mu{}^{ij}$, is given by (2.8*d*) with (2.11*a* and *b*). Thus, the symmetry-breaking parameter R appears as a natural dimensional constant which essentially sets the scale for the associated fundamental interaction (which for the gravitational coupling, is of the order of the Planck length $R = R_N = 10^{-35}$ m and for the short-range strong interaction should be of the order of one fermi: $R = R_S = 10^{-15}$ m). The first term is the gauge-invariant curvature scalar of $\mathfrak{M}_4$, the second term is the curvature kinetic energy of the gauge fields (the topological invariant) and the last term is the torsion kinetic energy. (That torsion has its origin in a dS SO(4, 1) gauge theory was already shown by Townsend [30].) The third term is the curvature scalar of the group SO(4, 1), the vacuum polarisation ('cosmological' constant), and sets the interaction scale.

As noted by Zardecki in [28], when the action is varied with respect to $\bar{\omega}_\mu{}^{ij}$ and $\bar{h}_\mu{}^i$, the following pair of field equations in the bundle $P'(\mathfrak{M}_4, \mathrm{SO}(3,1))$ is obtained:

$$\bar{D}_\mu[|g|^{1/2}\bar{R}^{ij\mu\nu} + (2/R^2)(\bar{h}^{i\mu}\bar{h}^{j\nu} - \bar{h}^{j\mu}\bar{h}^{i\nu})] + (2/R^2)|g|^{1/2}\bar{S}^{ji\nu} = 0 \qquad (2.16a)$$

$$\bar{D}_\mu[|g|^{1/2}\bar{S}^{i\mu\nu}] + |g|^{1/2}[\bar{R}^{i\nu} + (3/R^2)\bar{h}^{i\nu}] = 0 \qquad (2.16b)$$

where $\bar{D}_\mu$ denotes the covariant derivative in $P'(\mathfrak{M}_4, \mathrm{SO}(3,1))$. In the infinite symmetry breaking limit (when $R \to \infty$), SO(4, 1) goes over into ISO(3, 1) and the field equations supply both the Yang equation and Einstein's equation extended by torsion. Therefore the theory described by (2.16*a* and *b*) is equivalent to that of Einstein's gravity when the symmetry breaking parameter is very large and the constraint: $\bar{S}_\mu{}^{ij} = 0$ is imposed.

In this work our interest is not with the fundamental length associated to gravitation but lies in the interpretation of R as a naturally occurring length appropriate as the scaling parameter relevant to hadron physics where $G_S = R_S^2$ ($R_S = 10^{20}R_N$ with the Planck length $R_N = (G_N\hbar/c^3)^{1/2} = 2.0 \times 10^{-35}$ m). The typical Schwarzschild relation valid for hadronic dimensions reads:

$$\frac{2GM_p}{c^2} = R = \frac{\hbar}{M_p c}$$

where we have introduced the Compton wavelength of the proton, and one readily determines that $G = 10^{38}G_N = G_S$ which is of the order of the strong interaction. Therefore, in complete analogy with the gravitational coupling, the strong force should manifest itself through the explicit curvature of spacetime. We now reinterpret Einstein's cosmological constant (vacuum polarisation) as a new large 'cosmological' constant $\Lambda_S = (M_S c/\hbar)^2$. The corresponding field equations, being of the Einstein type, constitute a gauge theory of strong spin-2 interactions and the presence of the Λ_S term is equivalent to introducing a mass term in the associated Lagrangian. Therefore the space within the hadron becomes (strongly) curved and might cause, in the most symmetrical state (ground state configuration), a metric of the dS type where the associated Schwarzschild radius (Compton wavelength) acts as a natural confinement mechanism.

This allows one to view the structure of composite particles and hadron spectroscopy as originating from a purely geometrical theory of spacetime which finds complete justification in the context of strong gravity theories [31] invoked in the study of hadron physics [32-36] where the space within a hadron can be described by a metric of the ds type and the radius of this space is of the order of the range of the strong interaction. Beyond the radius of the interaction, the hadronic matter distribution falls rapidly to zero and a strictly flat spacetime is consequently established (in the absence of long-range gravitational perturbations). That is, the vanishing of the hadronic matter distribution is viewed as inducing a contraction of the ds group SO(4, 1) down to the flat space ISO(3, 1) group, the kinematical group for structureless elementary point particles embedded in M_4.

The usual notion of parallel transport of a Lorentz vector V^μ lying in $T_x(\mathfrak{M}_4)$ is generated by the covariant derivative:

$$\bar{D}_\mu = \partial_\mu + \mathrm{i}\tfrac{1}{2}\bar{\omega}_\mu{}^{ij}J_{ij}$$

by first converting the world components of V^μ into its nonlinear ones $\bar{V}^i$ using the vierbein field $\bar{V}^i = \bar{h}_\mu{}^i V^\mu$. Transport about an infinitesimal closed curve on spacetime results in the SO(3, 1) rotation

$$[\bar{D}_\mu, \bar{D}_\nu] = \mathrm{i}\tfrac{1}{2}\bar{R}_{\mu\nu}{}^{ij}J_{ij}$$

determined by the Riemann curvature tensor

$$\bar{R}_{\mu\nu}{}^{ij} = \partial_\mu\bar{\omega}_\nu{}^{ij} - \partial_\nu\bar{\omega}_\mu{}^{ij} + \bar{\omega}_\mu{}^{ik}\bar{\omega}_{\nu k}{}^{j} - \bar{\omega}_\nu{}^{ik}\bar{\omega}_{\mu k}{}^{j} \tag{2.17}$$

which represents the standard Yang-Mills field strength for the non-Abelian Lorentz group. Therefore the usual concept of parallel transport generated by the SO(4, 1) covariant derivative leads to the appearance of spacetime curvature only, thereby falling short of describing the complete geometrical picture of a ds SO(4, 1) gauge theory. And the underlying reason for this short-coming can be attributed to the absence of the appearance of a torsion term, the origin of which can be traced back to the fact that the (pseudo) translational gauge component does not take part in the typical transport process. In order to obtain the complete geometrical picture, we follow the work of Stelle and West in [8] and construct a second type of differential operator along with its associated notion of parallel transport (which in the present formulation is a (4+1) curved space analogue of the differential geometric process of development into a flat affine tangent space) in which the translational component turns out to play the central role.

3. The development process and gauge conditions

The generator of development specifies the horizontal direction and is defined in terms of the original linear gauge fields:

$$\Delta_\mu \equiv \partial_\mu + \mathrm{i}\tfrac{1}{2}\omega_\mu{}^{ij}J_{ij} - \mathrm{i}\theta_\mu{}^i\pi_i \tag{3.1}$$

and is a purely gauge-theoretic expression which essentially serves to map curves and vector fields defined on $\mathfrak{M}_4$ into their Σ^4 images. The idea is equivalent to that of rolling a copy of Σ^4 along a curve defined on $T_x(\mathfrak{M}_4)$ thereby transferring the curve and any associated vector field onto its surface. The presence of the linear gauge fields $\{\omega_\mu{}^{ij}, \theta_\mu{}^i\}$ is to ensure that Σ^4 will not 'slip' or 'twist' about the vertical as it rolls. We therefore interpret the arbitrariness in the choice of origin and coordinate axes orientation of Σ^4 as the freedom in making translational and rotational gauge choices.

Transformations induced by travelling about an infinitesimal closed path on spacetime are specified by the SO(4, 1) curvature fields

$$[\Delta_\mu, \Delta_\nu] = \mathrm{i}\tfrac{1}{2}Q_{\mu\nu}{}^{ij}J_{ij} - \mathrm{i}S_{\mu\nu}{}^i\pi_i \tag{3.2}$$

where the rotational and translational sectors have already been expressed in (2.8*c* and *d*). In the vacuum state $\{\omega_\mu{}^i, \theta_\mu{}^i\} = \{0, 0\}$ and spacetime is a dS space Σ^4 with $\Delta_\mu \equiv \partial_\mu$. That is, $\mathfrak{M}_4^{\mathrm{vac}}$ and Σ^4 fit together exactly and development involves no observable changes in the local frame at all. For this trivial gauge choice, the spin connection and vierbein field $\{\bar{\omega}_\mu{}^{ij}, \bar{h}_\mu{}^i\}$ are expressed as functions of the coset parameters and their derivatives only as given by (2.12*a* and *b*).

We now consider the purely geometric structure of spacetime by considering the action of Δ_μ on an arbitrary nonlinear vector field, $\bar{V}^i$, such that:

$$\Delta_\mu \bar{V}^i = (\partial_\mu + \mathrm{i}\tfrac{1}{2}\bar{\omega}_\mu{}^{ij}J_{ij} + \mathrm{i}\bar{h}_\mu{}^i\pi_i)\bar{V}^i = (\bar{D}_\mu + \mathrm{i}\bar{h}_\mu{}^i\pi_i)\bar{V}^i. \tag{3.3}$$

The meaning of this expression is the following (where we have used the notation of Stelle and West). First parallel transport the vector $\bar{V}^i(x+\mathrm{d}x)$ (lying on $T_{\xi(x+\mathrm{d}x)}(\Sigma^4_{x+\mathrm{d}x})$) across spacetime from $\mathrm{d}x^\mu$ to x^μ using $\bar{D}_\mu$ resulting in $\bar{V}^i(x)$ (lying in $T_{\xi(x)}(\Sigma^4_x)$) followed by an internal parallel transport in Σ^4_x (with help from the dS boosts, π_i) from $\xi^i(x)$ out to $\xi^i_*(x; x+\mathrm{d}x)$ (which lies in $T_{\xi_*(x+\mathrm{d}x)}(\Sigma^4_x)$). Therefore the process of development allows for the freedom in moving to various points in the internal dS fibre space attached to the point $x^\mu \in \mathfrak{M}_4$. (Note the change in sign of the translational term in (3.3) which is necessary for this interpretation and holds in the present formulation as a consequence of the SO(4, 1) Lie algebraic automorphism: $\pi_i \to -\pi_i$.) This description is simply a restatement of the fact that the gauge transformation induced by $\mathrm{i}\bar{h}_\mu{}^i\pi_i$ is completely equivalent to parallel transport in Σ^4_x generated by the dS space covariant derivative:

$$\overset{(0)}{D}_n = \partial_n + \mathrm{i}\tfrac{1}{2}\overset{(0)}{\omega}_n{}^{ij}J_{ij}$$

from $\xi^n(x) \in \Sigma^4_x$ out to $\xi^n_*(x; x+\mathrm{d}x) \in \Sigma^4_x$. The dS spin connection is $\overset{(0)}{\omega}_n{}^{ij} = \bar{h}^\mu{}_k \overset{(0)}{h}{}^k{}_n \bar{\omega}_\mu{}^{ij}$, where the dS vierbein field, $\overset{(0)}{h}{}_k{}^n$, serves to connect coordinate-induced (holonomic) indices n, r, s to the anholonomic i, j, k ones.

In analogy with the action of Δ_μ on a nonlinear vector field as expressed by (3.3), we have that the curvature terms as given by (2.8*a*–*d*) may be expressed according to:

$$\tfrac{1}{2}\mathfrak{R}_{\mu\nu}{}^{AB}J_{AB} \equiv \tfrac{1}{2}\bar{Q}_{\mu\nu}{}^{ij}J_{ij} + \bar{S}_{\mu\nu}{}^i\pi_i \tag{3.4a}$$

which has the geometrical interpretation that the dS curvature (rotational sector):

$$\bar{Q}_{\mu\nu}{}^{ij} = \bar{R}_{\mu\nu}{}^{ij} + R^{-2}(\bar{h}_\mu{}^i\bar{h}_\nu{}^j - \bar{h}_\nu{}^i\bar{h}_\mu{}^j) \tag{3.4b}$$

supplies the net SO(3, 1) rotation of the nonlinear components of the translated vector and is the difference between the usual Lorentz curvature tensor on spacetime (given by (2.17)) and the Lorentz curvature tensor of dS space. The torsion tensor (translational sector)

$$\bar{S}_{\mu\nu}{}^i = \partial_\mu\bar{h}_\nu{}^i - \partial_\nu\bar{h}_\mu{}^i + \bar{\omega}_\mu{}^i{}_k\bar{h}_\nu{}^k - \bar{\omega}_\nu{}^i{}_k\bar{h}_\mu{}^k \tag{3.4c}$$

indicates the amount by which an image curve on Σ_x^4, corresponding to a small closed curve on spacetime, fails to close. (The transported vector does not remain attached to the parallel congruence of geodesics but is rotated relative to nearby geodesics by the action of (3.4*c*).)

For the SO(4, 1) gauge theory, the mechanism of spontaneous symmetry breakdown is triggered by constraining the non-dynamical field $\Phi^A(x)$ to a (4+1) ds space with $\Phi^A\Phi_A = -R^2$. The local SO(4, 1) invariance can be used to orientate $\Phi^A(x)$ in some prescribed direction which breaks the full ds symmetry down to that of one of its Lorentz subgroups. For example, in the so-called 'unitary gauge' one sets

$$\Phi^A(x) = \delta_5{}^A R$$

so that from (2.2*c*)

$$\pi_i = R^{-1} J_{5i}$$

and from (2.5 and 2.7*c*)

$$\nabla_\mu \Phi^i(x) = R\Gamma_\mu{}^{5i}(x) = \theta_\mu{}^i \qquad \nabla_\mu \Phi^5(x) = 0. \tag{3.5}$$

That is, for the unitary gauge choice the Higgs field is constrained to point in the fifth direction and the corresponding coset parameters $\xi^i(x) = (0, 0, 0, 0)$ everywhere. Thus any possible reference to the Goldstone field and its derivative is eliminated from the Lagrangian and any trace that the $\xi^i(x)$ might play in a full ds SO(4, 1) gauge theory is entirely suppressed. So the unitary gauge demands that

$$\{\omega_\mu{}^{ij}, \theta_\mu{}^i\} = \{\bar\omega_\mu{}^{ij}, \bar h_\mu{}^i\}$$

(unitary gauge) where $\theta_\mu{}^i$ is identified with the spacetime vierbein field that provides the solder between $T_x(\mathfrak{M}_4)$ and $T_{\xi(x)}(\Sigma_x^4)$.

While the complete dynamical theory should be fully gauge-invariant, observations and the description of any specific physical model should be associated with the additional imposition of certain gauge-fixing conditions. These gauge constraints would lead to the vanishing of certain curvature fields (rotational and/or translational) contained in (3.4*a*). For example, if $\bar R_{\mu\nu}{}^{ij} = 0$ over some region in spacetime, then (adhering to the usual principle of classical gauge field theory) there should exist a horizontal cross section (gauge) with $\bar\omega_\mu{}^{ij} = 0$ (complete absence of SO(3, 1) gauge interactions). In this case, directions take on a global meaning in the region over $\bar R_{\mu\nu}{}^{ij} = 0$. In other words, trivialising the Lorentz gauge such that $\{\bar\omega_\mu{}^{ij}, \bar h_\mu{}^i\} = \{0, \bar h_\mu{}^i\}$ generates a ds (4+1) spacetime enhanced with torsion $\bar S_{\mu\nu}{}^i \neq 0$, and as the strength of the symmetry breaking tends to infinity ($R \to \infty$) the theory limits to the description of spacetime carrying a telleparallelism [37]:

$$\mathfrak{M}_4 \xrightarrow{R\to\infty} U_4 \xrightarrow{\bar R_{\mu\nu}{}^{ij}=0} T_4.$$

Furthermore, for a spacetime to be as flat as possible, we expect that both $\bar R_{\mu\nu}{}^{ij} = 0$ and $\bar S_{\mu\nu}{}^i = 0$ and the preferred cross section should be the one that follows the horizontal direction with $\{\bar\omega_\mu{}^{ij}, \bar h_\mu{}^i\} = \{0, 0\}$. However, classical gauge field theory suggests that if $\bar h_\mu{}^i = 0$ we then obtain the rather unsatisfactory result of no observable change in position (the solder between $T_x(\mathfrak{M}_4)$ and $T_{\xi(x)}(\Sigma_x^4)$ breaks down and we completely lose the rigid linkage between fibre and base space). In order to circumvent this difficulty we trivialise the ds gauge potentials in the unitary gauge by finding a certain class of Lorentz connected cross sections such that:

$$\{\bar\omega_\mu{}^{ij}, \bar h_\mu{}^i\} = \{0, \delta_\mu{}^i\}$$

(inertial gauge) and impose the *constraint* $\bar{S}_{\mu\nu}{}^{i}=0$ (a global sense of position) leading to:

$$\partial_\mu \delta_\nu{}^{i} - \partial_\nu \delta_\mu{}^{i} = 0 \tag{3.6a}$$

$$\bar{R}_{\mu\nu}{}^{ij} = 0 \tag{3.6b}$$

$$\bar{Q}_{\mu\nu}{}^{ij} = R^{-2}(\delta_\mu{}^{i}\delta_\nu{}^{j} - \delta_\nu{}^{i}\delta_\mu{}^{j}) \tag{3.6c}$$

which describes a maximally flat spacetime with:

$$[\Delta_\mu, \Delta_\nu] = \mathrm{i}\tfrac{1}{2}\bar{Q}_{\mu\nu}{}^{ij}J_{ij} = \mathrm{i}R^{-2}J_{\mu\nu}. \tag{3.7}$$

Therefore for a ds SO(4, 1) gauge theory the maximally flat spacetime is described by a non-flat connection thereby justifying its observable translational structure [38] and one cannot regard the underlying spacetime in the immediate neighbourhood of its associated ds fibre space to be strictly flat. With the ds gauge group, the observation of translations is directly attributed to the fact that, in the 'inertial frame', the Δ_μ are non-commuting for all possible gauge choices.

We close this section with a few remarks concerning the extended object's relativistic displacement and centre of mass vectors. In usual classical mechanics, one typically associates the motion of a structureless elementary point particle with the point to which the mass (energy due to internal excitations) of the object is attached. In that description, the object's centre coincides with the origin. For the non-local object we have a naturally occurring fundamental length which is taken to specify the object's non-trivial extension. Now the centre of the extended object becomes, in its most general configuration, displaced from the origin where the relativistic definition of the centre of mass is given by the Lorentz four-vector [39, 40]:

$$Y_\mu = J_{\nu\mu}\frac{P^\nu}{m^2} - \frac{P_\mu J_{\nu 0}P^\nu}{m^2 P^0} + \frac{P_\mu t}{P^0} = q_\mu + \hat{P}_\mu(t - q_0) \tag{3.8}$$

where the relativistic displacement:

$$q_\mu = J_{\nu\mu}\frac{P^\nu}{m^2} \tag{3.9}$$

specifies how much the object's geometrical centre is displaced from the origin when the velocity $\hat{P}_\mu = P_\mu(P^0)^{-1}$ vanishes, $\hat{P}_\mu = (1, 0, 0, 0)$. When the object is in motion, the vector q_μ signifies how far the centre of mass trajectory is displaced from the origin.

4. Quantisation and the relativistic Hamiltonian

So far we have treated the ds structure group and its SO(3, 1) stability subgroup as they pertain to a geometrical gauge theory of a non-local microphysical object in the soldered fibre bundle formalism. However, a quantum physical system is not described in purely geometrical terms but, according to the fundamental postulates of quantum mechanics, by an algebra of operators which act in the space of physical states. In particular, according to Wigner [41], one associates with the symmetry group of motion in Minkowski space, M_4, the physical states (structureless elementary point particles) which are described by the irreducible unitary representation spaces of the Poincaré group (the generators of which are represented by the Hermitian operators: momentum P_μ and total angular momentum $J_{\mu\nu} = Q_\mu P_\nu - Q_\nu P_\mu + S_{\mu\nu}$). In this paper we shall follow Wigner and use the representation spaces of ISO(3, 1) and the algebra of observables generated by P_μ and $J_{\mu\nu}$ as a basis for an SO(4, 1) gauge-theoretical description of extended single hadron systems at the quantised level.

In order to resolve the well known mass degeneracy (decoupled mass and spin) inherent in the Poincaré group description of structureless elementary point particles we introduce a symmetry breaking interaction by using a substitution analogous to that used in the electromagnetic interaction: the minimal coupling scheme. However, the required strength of the symmetry breaking must be on the order of the strong interaction thereby supplying a fundamental length leading to a description of extended hadronic structures of constant curvature with radius on the order of $R_S = 1$ fm.

Therefore we return to the purely gauge-theoretical SO(4, 1) process of development and, for the simplest case of non-interacting systems, employ the horizontal Lorentz cross section in the unitary gauge such that:

$$\Delta_\mu = \partial_\mu - \mathrm{i}\lambda J_{5\mu} \tag{4.1}$$

where we have used (3.1) and replaced the (geometrical) radius of Σ^4 with the corresponding strength of the symmetry breaking: $R^{-1} = \lambda \approx 1$ GeV. We now go over to the position representation where momentum $P_\mu = -\mathrm{i}\partial_\mu$ and, correspondingly, generalised momentum $B_\mu = -\mathrm{i}\Delta_\mu$ give:

$$B_\mu = P_\mu - \lambda J_{5\mu} \tag{4.2}$$

where $J_{5\mu}$ is naturally identified with the quantum mechanical analogue of the relativistic displacement of the system's origin, (3.9). Therefore,

$$J_{5\mu} \equiv \hat{b}_\mu (= b_\mu M) = \tfrac{1}{2}\{J_{\mu\rho}, \hat{P}^\rho\} \tag{4.3}$$

(where $P_\mu P^\mu = M^2$ and $\hat{P}_\mu = P_\mu M^{-1}$) which is, up to a sign, the dimensionless form of the quantum mechanical analogue of the relativistic displacement (Finkelstein [42] centre operator of the origin type). Therefore,

$$B_\mu = P_\mu - \lambda \hat{b}_\mu \tag{4.4}$$

which generates the motion (in a four-dimensional symmetrically curved dS space) corresponding to that generated by the P_μ (in a typical Minkowski space) and goes over into the motion generated by the P_μ in the limit of zero curvature (as the strength of the symmetry breaking tends to infinity). Clearly, this particular feature of dS spacetime is of little significance in the usual cosmological sense where $R \approx 10^{26}$ m which minimises any possible affects that the $J_{5\mu}$ might have on the ∂_μ in (4.1). However, in the context of the hadronic short-range field described by a metric of the dS type the situation changes since the associated radius is now of the order of 1 fm thereby disallowing the approximation of SO(4, 1) with ISO(3, 1). In this realm, one cannot ignore the contributions of the $\hat{b}_\mu$ in (4.4) which serve to introduce a dS space interaction resulting in a perturbation on the underlying quantum relativistic dynamics generated by P_μ and $J_{\mu\nu}$ and eventually lead to a non-degenerate mass–spin trajectory relation (as discussed below). It is as if the intrinsic curvature of the dS fibre space has removed the mass degeneracy of the Minkowski space description by supplying the necessary symmetry breaking interaction.

The relativistic displacement vector q_μ as expressed by (3.9) has been introduced by assuming that the extended object's centre of mass and origin do not necessarily coincide. Quantum mechanically we have that

$$-q_\mu{}^{QM} \equiv b_\mu (= \hat{b}_\mu M^{-1}) = \tfrac{1}{2} M^{-1}\{J_{\mu\rho}, \hat{P}^\rho\} \tag{4.5}$$

and, since the angular momentum consists of both orbital and intrinsic components:

$$b_\mu = \tfrac{1}{2}M^{-1}\{Q_\mu P_\rho - Q_\rho P_\mu + S_{\mu\rho}, \hat{P}^\rho\} = Q_\mu + \mathrm{i}\tfrac{3}{2}\hat{P}_\mu M^{-1} - (\hat{P}\cdot Q)\hat{P}_\mu + d_\mu$$

where

$$d_\mu = S_{\mu\rho}P^\rho \tag{4.6}$$

denotes the intrinsic part of b_μ such that

$$b_\mu = b_\mu^{\text{ext}} + b_\mu^{\text{int}}$$

with

$$b_\mu^{\text{ext}} = Q_\mu + \mathrm{i}\tfrac{3}{2}\hat{P}_\mu M^{-1} - (\hat{P}\cdot Q)\hat{P}_\mu \qquad b_\mu^{\text{int}} = d_\mu .$$

Therefore

$$Q_\mu = Y_\mu - d_\mu \tag{4.7}$$

where the quantum relativistic centre of mass:

$$Y_\mu = b_\mu + \hat{P}_\mu[(\hat{P}\cdot Q) - \mathrm{i}\tfrac{3}{2}M^{-1}].$$

However, in the (extended) particle's rest frame where $\hat{P}_\mu = (1, 0, 0, 0)$:

$$\hat{P}\cdot b = \mathrm{i}\tfrac{3}{2}M^{-1} = b_0$$

so that

$$Y_\mu = b_\mu + \hat{P}_\mu[(\hat{P}\cdot Q) - b_0].$$

In accordance with the non-quantum relativistic centre of mass expressed by (3.8) we now define proper time, $\tau \equiv (\hat{P}\cdot Q) - b_0$ and

$$Y_\mu = b_\mu + \hat{P}_\mu \tau. \tag{4.8}$$

The commutator of two d_μ's is:

$$[d_\mu, d_\nu] = -\mathrm{i}M^{-2}(S_{\mu\nu} + d_\nu P_\mu - d_\mu P_\nu)$$

which suggests the definition of the intrinsic spin tensor:

$$\Sigma_{\mu\nu} = S_{\mu\nu} - d_\mu P_\nu + d_\nu P_\mu \tag{4.9}$$

where

$$[\Sigma_{\mu\nu}, \Sigma_{\rho\sigma}] = -\mathrm{i}(\breve{g}_{\mu\rho}\Sigma_{\nu\sigma} + \breve{g}_{\nu\sigma}\Sigma_{\mu\rho} - \breve{g}_{\mu\sigma}\Sigma_{\nu\rho} - \breve{g}_{\nu\rho}\Sigma_{\mu\sigma})$$

with: $\breve{g}_{\mu\rho} = g_{\mu\rho} - \hat{P}_\mu\hat{P}_\rho$. From the definition (4.6) for d_μ and (4.9) for $\Sigma_{\mu\nu}$ we find the following operator identities:

$$d_\mu P^\mu = 0 \qquad \text{and} \qquad P^\mu \Sigma_{\mu\nu} = 0. \tag{4.10}$$

Inserting the definition for $\Sigma_{\mu\nu}$ into $J_{\mu\nu}$ allows the total angular momentum to be expressed as $J_{\mu\nu} = Y_\mu P_\nu - Y_\nu P_\mu + \Sigma_{\mu\nu}$ where the quantum relativistic centre of mass Y_μ is given by (4.8).

The quantum mechanical analogue of the purely gauge-theoretical process of development about a small closed curve in spacetime may be expressed according to:

$$[B_\mu, B_\nu] = \lambda^2[\hat{b}_\mu, \hat{b}_\nu] - \lambda(P_\mu\hat{b}_\nu - P_\nu\hat{b}_\mu) - \lambda(\hat{b}_\mu P_\nu - \hat{b}_\nu P_\mu).$$

Using the well known commutation relations of the Poincaré group with $\hat{P}_\mu = P_\mu M^{-1}$ and the dimensionless form of the quantum relativistic centre operator, $\hat{b}_\mu$ given in (4.5), we have that

$$[\hat{b}_\mu, P_\nu] = -\mathrm{i}(g_{\mu\nu} - \hat{P}_\mu \hat{P}_\nu)M \qquad \text{and} \qquad [P_\mu, \hat{b}_\nu] = \mathrm{i}(g_{\mu\nu} - \hat{P}_\mu \hat{P}_\nu)M.$$

This result leads to the identity

$$(P_\mu \hat{b}_\nu - P_\nu \hat{b}_\mu) - (\hat{b}_\mu P_\nu - \hat{b}_\nu P_\mu) = 0$$

which is recognised as the quantum mechanical equivalent of the geometrical constraint of (3.6*a*). Therefore we have that

$$[B_\mu, B_\nu] = \lambda^2[\hat{b}_\mu, \hat{b}_\nu] = \mathrm{i}\lambda^2 J_{\mu\nu} \tag{4.11a}$$

where, once again, we have used the expression for $\hat{b}_\mu$ and the commutation relations of the Poincaré group. The SO(4, 1) coupling constant λ determines the strength (domain) of the self-interaction (compensating) fields $\hat{b}_\mu$ and is geometrically related with the radius $R = \lambda^{-1}$ of the micro-dS space where R serves as a measure of the confinement distance (extension) of the isolated non-local object.

Equation (4.11*a*) together with

$$[J_{\mu\nu}, J_{\rho\sigma}] = -\mathrm{i}(g_{\mu\rho}J_{\nu\sigma} + g_{\nu\sigma}J_{\mu\rho} - g_{\mu\sigma}J_{\nu\rho} - g_{\nu\rho}J_{\mu\sigma}) \tag{4.11b}$$

and

$$[B_\mu, J_{\rho\sigma}] = -\mathrm{i}(g_{\mu\sigma}B_\rho - g_{\mu\rho}B_\sigma) \tag{4.11c}$$

show that B_μ and $J_{\mu\nu}$ generate another representation of a dS SO(4, 1). This representation, first introduced by Bohm [43], turns out to play the central role in the construction of the model of the dS QRR in that the non-local quantum system is characterised by the eigenvalues $\lambda^2\alpha^2$ of the second-order Casimir operator of this SO(4, 1):

$$\lambda^2 C_{\mathrm{II}} = B_\mu B^\mu - (\lambda^2/2)J_{\mu\nu}J^{\mu\nu} \overset{\text{irrep}}{=\!=\!=} \lambda^2\alpha^2 \tag{4.12}$$

in the same way that the quantum relativistic (structureless) point particle is characterised by the eigenvalues m^2 of the Poincaré invariant

$$P_\mu P^\mu \overset{\text{irrep}}{=\!=\!=} m^2c^2.$$

Substitution of (4.2) into (4.12) gives

$$\lambda^2 C_{\mathrm{II}} = P_\mu P^\mu + \tfrac{9}{4}\lambda^2 - \lambda^2 W(P_\mu P^\mu)^{-1} \overset{\text{irrep}}{=\!=\!=} \lambda^2\alpha^2 \tag{4.13}$$

where the spin operator $W(P_\mu P^\mu)^{-1} = \frac{1}{2}\Sigma_{\mu\nu}\Sigma^{\mu\nu}$ and $\Sigma_{\mu\nu}$ is the usual spin tensor which satisfies relation (4.9).

The quantum relativistic Hamiltonian is obtained by applying the methods of constrained Hamiltonian mechanics and the constraint for the structureless elementary point particle, $\Phi \equiv P_\mu P^\mu - m^2c^2 \approx 0$, is replaced with the constraint imposed on the second-order invariant of the dS SO(4, 1):

$$\Phi \equiv P_\mu P^\mu - \lambda^2 W(P_\mu P^\mu)^{-1} + \lambda^2(\tfrac{9}{4} - \alpha^2) \approx 0. \tag{4.14}$$

The symbol ≈ 0 signifies 'set weakly equal to zero' since the constraint has nonvanishing commutators and one must evaluate all commutation relations prior to imposing the constraint. Following the rules of constrained Hamiltonian mechanics, one obtains the following quantum relativistic Hamilton operator:

$$\mathfrak{H} = \phi\Phi \equiv \phi[P_\mu P^\mu - \lambda^2 W(P_\mu P^\mu)^{-1} + \lambda^2(\tfrac{9}{4} - \alpha^2)] \tag{4.15}$$

where ϕ is a velocity parameter (Lagrange multiplier) and in the time-like centre of mass gauge is determined to be (see Aldinger [23]) $\phi = -1/2M$. The constraint relation taken between the canonical basis vectors: $|p\, s\, s_3\rangle$ (which form a basis of the space of physical states [44]) leads to the mass-spin trajectory relation:

$$m^2 = \lambda^2(\alpha^2 - \tfrac{9}{4}) + \lambda^2 s(s+1) \tag{4.16}$$

where

$$W(P_\mu P^\mu)^{-1} = \tfrac{1}{2}(\Sigma_{\mu\nu}\Sigma^{\mu\nu}) \overset{\text{irrep}}{=\!=\!=} s(s+1).$$

The possible values for the SO(4, 1) eigenvalues, $\lambda^2\alpha^2$, are limited by restricting the α to the principal series representation [45] (only those that go over into the physical representations of the Poincaré group) where $\alpha^2 > \frac{9}{4} - s(s+1)$ thus demanding $m^2 > 0$ and no tachyonic states arise.

The phenomenological value for $\alpha^2_{\text{meson}} = \frac{9}{4}$ and for $\alpha^2_{\text{baryon}} = \frac{9}{2}$. An empirical value for the hadronic mass (coupling) constant λ may be determined from the fits of the experimental data and is found to be $\lambda = 0.53$ GeV which leads to a micro-ds space radius $R = 0.37 \times 10^{-15}$ m (see Aldinger *et al* [22] for a complete description of comparison to the experimental data).

In the infinite ds space radius limit, when the compensating gauge operators $\hat{b}_\mu$ are 'turned off', the generalised momenta:

$$B_\mu = P_\mu - \lambda \hat{b}_\mu \to P_\mu$$

and the ds SO(4, 1) gauge group contracts into ISO(3, 1). Moreover, in order to obtain a faithful representation in this contraction limit (as the strength of the symmetry breaking tends to infinity) one must go through a sequence of representations in such a way that $\lambda^2\alpha^2 \to m^2c^2 > 0$ which characterises the representations of the physical Poincaré group. Therefore the second-order invariant of SO(4, 1) reduces according to

$$P_\mu P^\mu + \tfrac{9}{4}\lambda^2 - \lambda^2 W(P_\mu P^\mu)^{-1} \overset{\text{irrep}}{=\!=\!=} \lambda^2\alpha^2 \to P_\mu P^\mu \overset{\text{irrep}}{=\!=\!=} m^2c^2$$

where the square of the momentum decouples from the spin and the SO(4, 1) Hamiltonian goes over into the familiar Hamiltonian of the structureless quantum relativistic mass point which is characterised by the kinematical Poincaré group.

Thus the curvature of spacetime breaks the mass degeneracy inherent in the flat space mass operator where the radius of the curvature plays the role of the symmetry breaking interaction. This result should be compared with general relativity which requires a spacetime curvature to arise from energies of various interactions.

If one wants to describe a tower of hadrons (as opposed to a single hadronic bound state), where each hadron is considered as a different state of the physical system 'hadron tower', one must take a reducible representation space of the Poincaré group and introduce operators which describe transitions between the different irreducible representations of ISO(3, 1). Therefore it cannot be constructed in terms of the Poincaré algebra and in analogy to the Dirac γ-matrices, which fulfil an analogous purpose for the theory of the electron, we choose a Hermitian vector operator Γ_μ which together with the intrinsic angular momentum $S_{\mu\nu}$ (generalisation of the $\sigma_{\mu\nu}$) form the simplest unitary (infinite-dimensional) representation of a ds SO(3, 2). The Γ_μ and $S_{\mu\nu}$ satisfy the commutation relations:

$$[S_{\mu\nu}, S_{\rho\sigma}] = -\mathrm{i}(g_{\mu\rho}S_{\nu\sigma} + g_{\nu\sigma}S_{\mu\rho} - g_{\mu\sigma}S_{\nu\rho} - g_{\nu\rho}S_{\mu\sigma}) \tag{4.17a}$$

$$[S_{\rho\sigma}, \Gamma_\mu] = \mathrm{i}(g_{\sigma\mu}\Gamma_\rho - g_{\rho\mu}\Gamma_\sigma) \tag{4.17b}$$

$$[\Gamma_\rho, \Gamma_\sigma] = -\mathrm{i}S_{\rho\sigma}. \tag{4.17c}$$

The additional representation fixing relation:

$$\{\Gamma_\rho, \Gamma_\sigma\} + \{S_{\rho\mu}, S_\sigma{}^\mu\} = -g_{\rho\sigma} \tag{4.18}$$

(Majorana representation) specifies, of the many irreducible representations of the commutation relations (4.17a–c), the four Majorana representations [46] whose main feature is that they contain only one irreducible representation of the $\mathrm{SO}(3,1)_{S_{\mu\nu}}$ subgroup. (Equation (4.18) is the analogue of the relation $\{\Gamma_\mu, \Gamma_\nu\} = \frac{1}{2}g_{\mu\nu}$ for the four-dimensional Dirac case.)

One of the many consequences of the representation fixing relation of (4.18) is that the spin operator:

$$W(P_\mu P^\mu)^{-1} = \tfrac{1}{2}\Sigma_{\mu\nu}\Sigma^{\mu\nu} = (\hat{P}\cdot\Gamma)^2 - \tfrac{1}{4} \tag{4.19}$$

which gives the following in an irreducible representation:

$$W(P_\mu P^\mu)^{-1} \overset{\text{irrep}}{=\!=\!=} (s+\tfrac{1}{2})^2 - \tfrac{1}{4} = s(s+1) \tag{4.20}$$

where spin-$s = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$. Therefore, with the substitution of (4.20) into (4.15) we may express the quantum relativistic Hamiltonian valid in the Majorana representation of SO(3, 2) as:

$$\mathfrak{H}^{\mathrm{Maj}} = -\frac{1}{2M}[P_\mu P^\mu - \lambda^2(\hat{P}\cdot\Gamma)^2 + \lambda^2(\tfrac{5}{2} - \alpha^2)] \tag{4.21}$$

which shall be used in the following section to determine the system's dynamical equations of motion.

5. Quantum relativistic dynamics

In order to obtain the time derivatives of the physical observables, $\mathfrak{L}$, the quantum analogue of constrained Hamiltonian mechanics is used and, therefore, the derivatives with respect to the evolution parameter τ (proper time) are evaluated using $\mathrm{d}\mathfrak{L}/\mathrm{d}\tau = \dot{\mathfrak{L}} = -\mathrm{i}[\mathfrak{L}, \mathfrak{H}^{\mathrm{Maj}}]$ prior to imposing the constraint of (4.14). Using (4.21) for the relativistic Hamiltonian, the following τ derivatives are obtained:

$$\dot{\hat{d}}_\mu = \dot{S}_{\mu\rho}\hat{P}^\rho = \tfrac{1}{2}\lambda^2\{\hat{P}\cdot\Gamma, \Gamma_\mu - (\hat{P}\cdot\Gamma)\hat{P}_\mu\}M^{-1} \tag{5.1a}$$

(where the fact that $\dot{\hat{P}}_\mu = 0$ has been used),

$$\dot{\Gamma}_\mu = -\tfrac{1}{2}\lambda^2\{\hat{P}\cdot\Gamma, \hat{d}_\mu\}M^{-1} \tag{5.1b}$$

and

$$\dot{Q}_\mu = \hat{P}_\mu - \lambda^2(\hat{P}\cdot\Gamma)\Gamma_\mu M^{-2} + \lambda^2(\hat{P}\cdot\Gamma)^2\hat{P}_\mu M^{-2} + \mathrm{i}\tfrac{1}{2}\lambda^2\hat{d}_\mu M^{-2}. \tag{5.1c}$$

An explicit τ-dependent expression for the particle position $Q_\mu(\tau)$ may be obtained by directly integrating (5.1c) leading to

$$Q_\mu(\tau) = \hat{P}_\mu\tau - \lambda^2(\hat{P}^\rho\Gamma_\rho)M^{-2}\int \Gamma_\mu(\tau)\,\mathrm{d}\tau + \lambda^2(\hat{P}^\rho\Gamma_\rho)^2M^{-2}\hat{P}_\mu\tau$$

$$+\mathrm{i}\tfrac{1}{2}\lambda^2M^{-2}\int \hat{d}_\mu(\tau)\,\mathrm{d}\tau + D_\mu \tag{5.2}$$

where D_μ is a τ-independent constant of integration. Furthermore, we have that

$$\ddot{\Gamma}_\mu(\tau) = -\lambda^2(\hat{P}^\rho\Gamma_\rho)M^{-1}\dot{\hat{d}}_\mu(\tau) - \mathrm{i}\tfrac{1}{2}\lambda^2 M^{-1}\dot{\Gamma}_\mu(\tau)$$

and

$$\ddot{\hat{d}}_\mu(\tau) = \lambda^2(\hat{P}^\rho\Gamma_\rho)M^{-1}\dot{\Gamma}_\mu(\tau) - \mathrm{i}\tfrac{1}{2}\lambda^2 M^{-1}\dot{\hat{d}}_\mu(\tau)$$

the solutions to which may be expressed as:

$$\Gamma_\mu(\tau) = (\hat{P}^\rho\Gamma_\rho)\hat{P}_\mu - \exp\left(-\mathrm{i}\frac{\lambda^2}{2M}[1+2(\hat{P}^\rho\Gamma_\rho)]\tau\right)\hat{A}_\mu - \exp\left(\mathrm{i}\frac{\lambda^2}{2M}[-1+2(\hat{P}^\rho\Gamma_\rho)]\tau\right)\hat{A}^\dagger_\mu \tag{5.3a}$$

$$\hat{d}_\mu(\tau) = -\mathrm{i}\exp\left(-\mathrm{i}\frac{\lambda^2}{2M}[1+2(\hat{P}^\rho\Gamma_\rho)]\tau\right)\hat{A}_\mu + \mathrm{i}\exp\left(\mathrm{i}\frac{\lambda^2}{2M}[-1+2(\hat{P}^\rho\Gamma_\rho)]\tau\right)\hat{A}^\dagger_\mu \tag{5.3b}$$

where the fundamental mode operators (see Aldinger [23])

$$\hat{A}_\mu = \hat{A}^x_\mu + \mathrm{i}\hat{A}^y_\mu \tag{5.4a}$$

with

$$\hat{A}^x_\mu \equiv \frac{(\hat{P}^\rho\Gamma_\rho)\hat{P}_\mu}{2} - \frac{\Gamma_\mu(0)}{2} \tag{5.4b}$$

$$\hat{A}^y_\mu \equiv \frac{\hat{d}_\mu(0)}{2}. \tag{5.4c}$$

Substituting (5.3*a* and *b*) into (5.2) yields:

$$Q_\mu(\tau) = b_\mu + \hat{P}_\mu\tau - d_\mu(\tau) \tag{5.5}$$

where the term $b_\mu + \hat{P}_\mu\tau$ describes the collective motion of the centre of mass of the non-local object with centre of mass

$$Y_\mu = b_\mu + \hat{P}_\mu\tau$$

already given in (4.8). The other term, which can be expressed as

$$d_\mu = -\frac{\mathrm{i}}{M}\exp\left(-\mathrm{i}\frac{\lambda^2}{2M}\tau\right)\sum_{n=-\infty}^{\infty}\left[\exp\left(-\mathrm{i}n\frac{\lambda^2}{M}(\hat{P}^\rho\Gamma_\rho)\tau\right)\right]n^{-1}\hat{A}_{\mu n} \tag{5.6}$$

where we must restrict the modes to $n=\pm1$ and $\hat{A}^\dagger_{\mu 1} = \hat{A}_{\mu -1}$, describes the system's internal motions.

Equation (5.5) with (5.6) should be compared with the general solution to the particle position equation of motion for the quantum relativistic string [47–50] where

$$x^\mu(\sigma,\tau) = Q^\mu + \frac{P^\mu}{\pi}\tau + \frac{\mathrm{i}}{\sqrt{\pi}}\sum_{n=-\infty}^{\infty}\frac{\alpha^\mu_n}{n}\mathrm{e}^{-\mathrm{i}n\tau}\cos n\sigma$$

where π is a constant with dimension $(\text{mass})^2$, when the string's excitations are restricted to the lowest mode ($n=\pm1$) and one considers the dynamics at one endpoint only, i.e. $\sigma = 0$ where $x_\mu(\sigma, \tau)$ characterises the individual positions along the string's surface with $\tau_1 \leqslant \tau \leqslant \tau_2$ and $0 \leqslant \sigma \leqslant L$, and where L is the string's length.

The time derivative of total angular momentum is:

$$\dot{J}_{\mu\nu} = \dot{\hat{Y}}_\mu \hat{P}_\nu - \dot{\hat{Y}}_\nu \hat{P}_\mu + \dot{\Sigma}_{\mu\nu} = \dot{Y}_\mu P_\nu - \dot{Y}_\nu P_\mu + \dot{\Sigma}_{\mu\nu}$$

where

$$\dot{\Sigma}_{\mu\nu} = 0 \qquad \text{and} \qquad \dot{J}_{\mu\nu} = 0.$$

Therefore $\dot{Y}_\mu$ must be parallel to P_μ (and that $\dot{\hat{Y}}_\mu$ must be parallel to $\hat{P}_\mu$). This, then, establishes the *Zitterbewegung*: as τ proceeds, the expectation value $\langle |Y_\mu|\rangle$ of the centre of mass operator follows a straight worldline in a direction parallel to $\langle |P_\mu|\rangle$, and the particle position $\langle |Q_\mu|\rangle = \langle |Y_\mu|\rangle - \langle |d_\mu|\rangle$ performs a helical motion about this worldline with a rotational frequency according to (5.6) given by

$$\omega_0 = \frac{\lambda^2}{2M}[1 \pm 2(\hat{P}^\rho \Gamma_\rho)]$$

which, for R on the order of one fermi and using the proton mass M_p, gives a frequency of the order of 10^{23} Hz. In the infinite symmetry breaking limit, $\dot{\hat{d}}$ and $\ddot{\hat{d}}$ go to zero. That is, a relativistic mass point does not perform *Zitterbewegung*. Thus, it is the *Zitterbewegung* that causes the hadronic 'size' and modifies the affine (Poincaré) description for the extensionless rotator into that of the dS description which is a dS space of radius R on the order of a fermi, i.e. an extended relativistic object.

We can also obtain an idea for the 'size' of the QRR, i.e. the radius of the spiral given by: $\sqrt{\vec{d}^2}$. One may directly verify that in the Majorana representation:

$$\hat{d}_\mu \hat{d}^\mu = -\tfrac{3}{4} - \hat{W} = -\tfrac{1}{2} - (\hat{P}^\rho \Gamma_\rho)^2. \tag{5.7}$$

This expression taken between rest states $|p = 0, s\, s_3\rangle$, leads to

$$d^2 = \langle s_3\, s\, 0| - d_\mu d^\mu |0, s\, s_3\rangle = \langle s_3\, s\, 0|\boldsymbol{d}^2|0, s\, s_3\rangle$$

and has the spectrum

$$d^2 = [\tfrac{3}{4} + s(s+1)]\frac{1}{m^2}.$$

With the mass formula (4.15), one obtains

$$d = \frac{1}{\lambda}\left(\frac{1}{1 + \{(\alpha^2 - 3)/[(s + \frac{1}{2})^2 + \frac{1}{2}]\}}\right)^{1/2} \to \frac{1}{\lambda}$$

which is seen to be on the order of $1/\lambda$ and approaches $1/\lambda$ for large values of s. $1/\lambda = R$ was the radius of the dS space in which SO(4, 1) acts as the symmetry group of motion. Furthermore, from (5.7) we have that $d^2 =$ constant leading to the description of a rigid rotating system. (That is, for the special case of the Majorana representation we have a simplified model in which there is no way to incorporate the fine structure effects due to centrifugal stretching and in order to consider these effects, one would have to go to a more general representation of SO(3, 2)).

The commutation relations of the relative (internal) variables with the system's centre of mass momentum:

$$[P_\mu, d_\nu] = 0 = [P_\mu, \dot{d}_\nu]$$

display the fact that the relative variables are translationally invariant while

$$[b_\mu, d_\nu] = \mathrm{i} d_\mu P_\nu M^{-2} \qquad [b_\mu, \dot{d}_\nu] = \mathrm{i}\dot{d}_\mu P_\nu M^{-2}$$

show a non-trivial 'mixing' between the relative (internal) position and the system's spacetime (external) centre operator of the origin type.

From the orthogonality relation. $\hat{P}^{\rho}d_{\rho}=0$, and the explicit form of d_{μ} given by (5.3*b*), we have that $\hat{P}\cdot\hat{A}=\hat{P}\cdot\hat{A}^{\dagger}=0$ (i.e. the fundamental mode operators are space-like four-vectors). Therefore for the model of the QRR we have an orthogonality relation which leads to an operator identity ensuring the elimination of ghost states. This method of ghost elimination is reminiscent to that of the covariant non-canonical relativistic string in the centre of mass gauge [49] where $P\cdot\xi=0$ (which implies $P\cdot a_n=0$ for $n>0$) guarantees a positive definite Hilbert space.

The dS QRR mode operators satisfy the following set of commutation relations:

$$[\hat{A}_{\mu},\hat{A}_{\nu}]=[\hat{A}^{\dagger}_{\mu},\hat{A}^{\dagger}_{\nu}]=0$$
$$[\hat{A}_{\mu},\hat{A}^{\dagger}_{\nu}]=-\tfrac{1}{2}(g_{\mu\nu}-\hat{P}_{\mu}\hat{P}_{\nu})(\hat{P}^{\rho}\Gamma_{\rho})-\tfrac{1}{2}\mathrm{i}\Sigma_{\mu\nu}$$

which does not agree with the covariant canonical or, for that matter, non-canonical algebra of the quantum relativistic string for which:

$$[a^{\mu}_{m},a^{\nu}_{n}]=0=[a^{\mu\dagger}_{m},a^{\nu\dagger}_{n}]$$
$$[a^{\mu}_{m},a^{\nu\dagger}_{n}]=(g^{\mu\nu}-\hat{P}^{\mu}\hat{P}^{\nu})\delta_{m,n}\qquad (m,n>0).$$

Therefore one must conclude that the dS QRR mode operators are not ordinary operators of the harmonic oscillator type. However, they do satisfy the following relations (where $\hat{A}^{z}\equiv\hat{P}^{\rho}\Gamma_{\rho}$):

$$[\hat{A}^{z},\hat{A}_{\mu}]=-\hat{A}_{\mu}$$
$$[\hat{A}^{z},\hat{A}^{\dagger}_{\mu}]=\hat{A}^{\dagger}_{\mu}$$
$$[\hat{A}_{\mu},\hat{A}^{\mu\dagger}]=-\tfrac{3}{2}\hat{A}^{z}$$

and

$$[\hat{A}^{2},\hat{A}_{\mu}]=0=[\hat{A}^{2},\hat{A}^{\dagger}_{\mu}]$$

where

$$\hat{A}^{2}=(\hat{A}^{x}_{\mu})^{2}+(\hat{A}^{y}_{\mu})^{2}+(\hat{A}^{z}_{\mu})^{2}=\tfrac{1}{2}(\hat{P}^{\rho}\Gamma_{\rho})^{2}-\tfrac{1}{4}$$

and $(\hat{P}\cdot\Gamma)$ has the spectrum n ($=s+\frac{1}{2}$ for the special case of the Majorana representation used here). Therefore the basic action of the QRR mode operators is to raise and lower spin for this particular representation and should, therefore, be considered as new relativistic analogues of the usual ladder operators of angular momentum.

6. Conclusions

A geometrical SO(4, 1) gauge theory for a collective model description of relativistically extended hadronic bound states, valid in the strongly-interacting regime of QCD, is established by considering a dS fibre bundle with Cartan connection. The full dS symmetry is broken down to that of its SO(3, 1) stability subgroup leading to a non-linear realisation of SO(4, 1) on the homogeneous non-compact coset space $\Sigma^4=$ SO(4, 1)/SO(3, 1), the closed dS universe. The symmetry breaking parameter is the radius of Σ^4 where the direction of the breaking is specified by the usual Higgs-type mechanism. The resulting Goldstonians are taken to represent coordinates of a point in Σ^4 and are used, along with the original linear gauge fields, to define the (short-range) vierbein and spin connection. The usual concept of parallel transport across spacetime is generated by using the covariant derivative defined in terms of the spin connection leading to the (short-range) SO(3, 1) curvature field.

We write down a Lagrangian that is quadratic in the curvature of the Cartan connection leading, after pullback, to a system of coupled equations for curvature and torsion in $P'(\mathfrak{M}_4, \mathrm{SO}(3,1))$. Under contraction of SO(4, 1) to ISO(3, 1) (when the dS length is large) the equations become Yang's equation and Einstein's equation extended by torsion. We interpret the dS symmetry breaking parameter as a fundamental length appropriate as the scaling parameter relevant to hadron physics thereby introducing a new large 'cosmological constant' (vacuum polarisation) in the resulting relativistic field equations. Therefore the space within a hadron becomes strongly curved and might cause, in the most symmetrical state, a metric of the dS type where the associated Schwarzschild radius (Compton wavelength) acts as a natural confinement mechanism.

In order to arrive at the complete geometrical picture entailing the effects of both spacetime curvature and torsion while retaining the full Yang-Mills gauge-theoretic concept throughout, we introduce a generalised covariant derivative along with its associated notion of parallel transport which is a SO(4, 1) (curved space) generalisation of the process known from differential geometry as development into the flat affine tangent space of a differentiable manifold. The generator of the development process is defined in terms of the linear gauge fields (which generate pseudo translations and rotations in Σ^4) and is a purely gauge-theoretic expression which serves to map curves and vector fields in spacetime into their Σ^4 images. The procedure essentially allows one to interpret the local choice of origin and coordinate axes orientation of Σ^4 as the freedom in making pseudo translational and rotational gauge choices. Geometrically, development about a small closed curve in spacetime leads to the proper interpretation of spacetime curvature and torsion.

The usual gauge concept of particle interactions for a relativistic theory (the minimal coupling scheme) is employed by interpreting the quantised SO(4, 1) generator of development as the SO(4, 1) 'generalised momentum' in which the pseudo translational piece supplies an experimentally detectable perturbation (hadron extension) to the conventional Minkowski description and gives some indication as to how much the extended microstructure's centre is displaced from the origin. (We have transformed from a passive Minkowski background without an interaction (arena for structureless elementary point particles) to a dynamical curved-space that forms an inseparable part of the non-local hadronic bound state.)

The 'generalised momenta' (under the horizontal Lorentz cross section in the unitary gauge choice) together with the SO(3, 1) generators of angular momentum supply another representation of a dS SO(4, 1) which forms the central feature in a quantum relativistic interpretation of an SO(4, 1) gauge theory in that the non-local object is now characterised by the eigenvalues of the second-order SO(4, 1) invariant operator in the same way that the structureless elementary point particle (without interaction) is characterised by the eigenvalues of the Poincaré invariant $P_\mu P^\mu \overset{\text{irrep}}{=\!=\!=} m^2c^2$. Using the usual rules of Dirac's constrained Hamiltonian mechanics, the second-order SO(4, 1) invariant is found to supply an experimentally verifiable mass–spin trajectory relation of the form $m^2 = m_0^2 + \lambda^2 s(s+1)$, thereby resolving the well known mass degeneracy inherent in the Poincaré group approach. The strong gravitational constant, $\lambda = R^{-1}$, represents the strength of the symmetry breaking interaction and from the known experimental data $\lambda = 0.53$ GeV leading to a micro-dS space radius (hadron extension) of $R = 0.37 \times 10^{-15}$ m.

In order to describe a 'tower' of hadrons where each hadron resonance is considered as a different state of the 'hadron tower' (physical system describing the quantum relativistic rotator, QRR), we introduce a Hermitian spin-changing vector operator which together with the intrinsic angular momentum forms the algebra of a dS SO(3, 2), the system's relativistic spectrum generating group. The special class of Majorana representations of SO(3, 2) leads to a rigid rotator model (no elasticity) and a relativistic Hamiltonian which is used to determine a completely solvable set of dynamical equations of motion resulting in standard features of extended object dynamics such as the *Zitterbewegung* with a calculated frequency on the order 10^{23} Hz.

Acknowledgments

The author is grateful to Professors A Bohm, L C Biedenharn, and H D Doebner for many useful conversations and suggestions concerning different aspects of this work.

References

[1] Kibble T W 1966 *J. Math. Phys.* **2** 212-21
[2] Sciama D W 1962 *Recent Developments in General Relativity* (Oxford: Pergamon) p 415
[3] Cho Y M 1976 *Phys. Rev.* D **14** 2521-5
[4] Hehl F W, von der Heyde P, Kerlick G D and Nester J M 1976 *Rev. Mod. Phys.* **48** 393-416
[5] Carmeli M 1977 *Group Theory and General Relativity* (New York: McGraw-Hill)
[6] D'Adda A 1982 *Theory of Fundamental Interactions* (Amsterdam: North-Holland) p 268
[7] Stelle K S and West P C 1979 *J. Phys. A: Math. Gen.* **8** L205-10
[8] Stelle K S and West P C 1980 *Phys. Rev.* D **21** 1466-88
[9] Tseytlin A A 1982 *Phys. Rev.* D **26** 3327-41
[10] Ivanenko D and Sardanashvily G 1983 *Phys. Rep.* **94** 1-45
[11] Ivanov E A and Niederele J 1982 *Phys. Rev.* D **25** 976-87
[12] Kobayashi S 1955 *Can. J. Math.* **8** 145-56
[13] Drechsler W 1975 *Fortschr. Phys.* **23** 607-48
[14] Drechsler W 1977 *Fiber Bundle Techniques in Gauge Theories, Lecture Notes in Physics* vol 67 (Berlin: Springer) p 145
[15] Coleman S, Wess J and Zumino B 1969 *Phys. Rev.* **177** 2239-47
[16] Callan C G, Coleman S, Wess J and Zumino B 1969 *Phys. Rev.* **177** 2247-50
[17] Salam A and Strathdee J 1969 *Phys. Rev.* **184** 1750-68
[18] Gürsey F and Marchildon L 1978 *Phys. Rev.* D **17** 2038-47
[19] Chang L N and Mansouri F 1978 *Phys. Rev.* D **17** 3168-78
[20] Kobayashi S and Nomizu K 1963 *Foundations of Differential Geometry* vol I (New York: Interscience)
[21] Aldinger R R *et al* 1983 *Phys. Rev.* D **28** 3020-31
[22] Aldinger R R *et al* 1984 *Phys. Rev.* D **29** 2828-37
[23] Aldinger R R 1985 *Phys. Rev.* D **32** 1503-11
[24] Aldinger R R 1986 *Int. J. Theor. Phys.* **25** 527-44
[25] Aldinger R R *et al* 1988 *Dynamical Groups and Spectrum-Generating Algebras* vol 2, ed A Bohm, Y Ne'eman and A O Barut (Singapore: World Scientific) p 773
[26] Dirac P A M 1964 *Lectures on Quantum Mechanics* (New York: Academic)
[27] Barut A O and Raczka R 1986 *Theory of Group Rep. and Applications* (Singapore: World Scientific) p 50
[28] Zardecki A 1988 *J. Math. Phys.* **29** 1661-6
[29] Mckellar R J 1984 *J. Math. Phys.* **25** 161-6
[30] Townsend P K 1977 *Phys. Rev.* D **15** 2795-801
[31] Isham C J, Salam A and Strathdee J 1971 *Phys. Rev.* D **3** 867-73
[32] Sivaram C and Sinha K P 1979 *Phys. Rep.* **51** 111-87
[33] Raut R and Sinha K P 1981 *Int. J. Theor. Phys.* **20** 69-77
[34] Chela-Flores J and Varela V 1983 *Phys. Rev.* D **27** 1248-53

[35] Sinha K P 1984 *Pramana* **23** 205–14
[36] de Sabbata V 1989 *Nuovo Cimento* **101A** 273–82
[37] Hehl F W 1985 *Found. Phys.* **15** 451–71
[38] Smrz P K 1980 *Found. Phys.* **10** 267–80
[39] Pryce M H L 1948 *Proc. R. Soc.* A **195** 62–81
[40] Umezawa M 1983 *J. Math. Phys.* **24** 2348–65
[41] Wigner E P 1939 *Ann. Math.* **40** 149–78
[42] Finkelstein R J 1949 *Phys. Rev.* **75** 1079–87
[43] Bohm A 1966 *Phys. Rev.* **145** 1212–8
[44] Bohm A *et al* 1983 *Phys. Rev.* D **28** 3032–40
[45] Bohm A 1975 *Studies in Mathematical Physics* ed A O Barut (Boston: Reidel) p 197
[46] Jaffe L 1971 *J. Math. Phys.* 882–91
[47] Scherk J 1975 *Rev. Mod. Phys.* **47** 123–64
[48] Schwarz J H 1982 *Phys. Rep.* **89** 223–322
[49] Rohrlich F 1976 *Nucl. Phys.* B **112** 177–84
[50] Almond D J 1983 *J. Phys. G: Nucl. Phys.* **9** 1309–45